Fano manifolds and blow-ups of low-dimensional subvarieties

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Abstract. We study Fano manifolds of pseudoindex greater than one and dimension greater than five, which are blow-ups of smooth varieties along smooth centers of dimension equal to the pseudoindex of the manifold.
We obtain a classification of the possible cones of curves of these manifolds, and we prove that there is only one such manifold without a fiber type elementary contraction.

1. Introduction

A smooth complex projective variety X is called Fano if its anticanonical bundle $-K_X$ is ample; the index r_X of X is the largest natural number m such that $-K_X = mH$ for some (ample) divisor H on X, while the pseudoindex i_X is the minimum anticanonical degree of rational curves on X.

By the Cone Theorem the cone $\mathrm{NE}(X)$ generated by the numerical classes of irreducible curves on a Fano manifold X is polyhedral. By the Contraction Theorem to each extremal ray of $\mathrm{NE}(X)$ is associated a contraction, i.e. a proper morphism with connected fibers onto a normal variety.
A natural question which arises from the study of Fano manifolds is to investigate - and possibly classify - Fano manifolds which admit an extremal contraction with special features: for example, this has been done in many cases in which the contraction is a projective bundle [**22, 21, 20, 23, 1, 18**], a quadric bundle [**28**] or a scroll [**5, 16**].

Recently, Bonavero, Campana and Wiśniewski have considered the case where an extremal contraction of X is the blow-up of a smooth variety along a point, giving a complete classification [**8**]. The case where the center of the blow-up is a curve has shown to be much more complicated. A complete classification in case $i_X \geq 2$ has been obtained in [**4**], following a more general theorem, where the classification of Fano manifolds with a contraction which is the blow-up of a manifold along a smooth subvariety of dimension $\leq i_X - 1$ is achieved. As for Fano

manifolds of pseudoindex $i_X = 1$ which are blow-ups of smooth varieties along a smooth curve, some special cases have been dealt with in the PhD thesis of Tsukioka [**25**] (partially published in [**24**]).

Considering the case when the dimension of the center of the blow-up is $i_X \geq 2$, the lowest possible dimension of the manifold is five; the cones of curves of such varieties are among those listed in the in [**11**], where the cone of curves of Fano manifolds of dimension five and pseudoindex greater than one were classified. Under the stronger assumption that $r_X \geq 2$ the complete list of Fano fivefolds which are blow-ups of smooth varieties along smooth surfaces has been given in [**12**].

In this paper we propose a generalization of both the results in [**4**] and in [**12**], considering Fano manifolds of dimension greater than five with a contraction which is the blow-up of a manifold along a smooth subvariety of dimension $i_X \geq 2$.
We will first give a classification of the possible cones of curves of these varieties:

THEOREM 1.1. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, with a contraction $\sigma\colon X \to Y$, associated to an extremal ray R_σ, which is a smooth blow-up with center a smooth subvariety B of dimension $\dim B = i_X$.*
Then the possible cone of curves of X are listed in the following table, where F stands for a fiber type contraction and D_{n-3} for the blow-up of a smooth variety along a smooth subvariety of codimension three.

ρ_X	i_X	R_1	R_2	R_3	R_4	
2		R_σ	F			*(a)*
2		R_σ	D_{n-3}			*(b)*
3	*2,3*	R_σ	F	F		*(c)*
3	*2*	R_σ	F	D_{n-3}		*(d)*
4	*2*	R_σ	F	F	F	*(e)*

We will then prove that there is only one Fano manifold satisfying the assumption of Theorem 1.1 whose cone of curves is as in case (b) - or, equivalently, which does not admit a fiber type contraction:

THEOREM 1.2. *Let X be a Fano manifold of dimension $n \geq 6$ and pseudoindex $i_X \geq 2$, which is the blow-up of another Fano manifold Y along a smooth subvariety B of dimension i_X; assume that X does not admit a fiber type contraction.*
Then $Y \simeq \mathbb{G}(1,4)$ and B is a plane of bidegree $(0,1)$.

We note that, in view of the classification given in Theorem 1.1 Generalized Mukai conjecture [**9, 2**] holds for the Fano manifolds we are considering.

Let us point out that the assumption $i_X \geq 2$ is essential for our methods, as well as for the ones used in [**4**], [**11**] and [**12**], on which they are based.

The proofs of Theorems 1.1 and 1.2 are contained in section 5 and 6. In section five we consider manifolds which possess a quasi-unsplit dominating family, proving that they are as in Theorem 1.1, cases (a) and (c)-(e).
In section six we consider manifolds which do not possess a family as above, proving first that their cone of curves is as in case (b), and then that the only manifold is the blow-up of $\mathbb{G}(1,4)$ along a plane of bidegree $(0,1)$.

2. Background material

2.1. Fano-Mori contractions. Let X be a smooth Fano variety of dimension n and let K_X be its canonical divisor. By Mori's Cone Theorem the cone $\mathrm{NE}(X)$ of effective 1-cycles, which is contained in the $\mathbb{R}$-vector space $N_1(X)$ of 1-cyles modulo numerical equivalence, is polyhedral; a face τ of $\mathrm{NE}(X)$ is called an extremal face and an extremal face of dimension one is called an extremal ray.
To every extremal face τ one can associate a morphism $\varphi : X \to Z$ with connected fibers onto a normal variety; the morphism φ contracts those curves whose numerical class lies in τ, and is usually called the Fano-Mori contraction (or the extremal contraction) associated to the face τ. A Cartier divisor D such that $D = \varphi^*A$ for an ample divisor A on Z is called a supporting divisor of the map φ (or of the face τ).
An extremal ray R is called numerically effective, or of fiber type, if $\dim Z < \dim X$, otherwise the ray is non nef or birational; the terminology is due to the fact that if R is non nef there exists an irreducible divisor D_R which is negative on curves in R. We usually denote with $E = E(\varphi) := \{x \in X \mid \dim \varphi^{-1}(\varphi(x)) > 0\}$ the exceptional locus of φ; if φ is of fiber type then of course $E = X$. If the exceptional locus of a birational ray R has codimension one, the ray and the associated contraction are called divisorial, otherwise they are called small.

2.2. Families of rational curves. For this subsection our main reference is [**15**], with which our notation is coherent; for missing proofs and details see also [**2**] and [**11**].
Let X be a normal projective variety and let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f\colon \mathbb{P}^1 \to X$; we consider the open subscheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$, corresponding to those morphisms which are birational onto their image, and its normalization $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$; the group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.1. The space $\mathrm{Ratcurves}^n(X)$ is the quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\mathrm{Aut}(\mathbb{P}^1)$; we define a family of rational curves to be an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$.
Given a rational curve $f \colon \mathbb{P}^1 \to X$ we will call a family of deformations of f any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of f.

Given a family V of rational curves, we have the following basic diagram, where p is a $\mathbb{P}^1$-bundle induced by the projection $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and i is the map induced by the evaluation $ev \colon \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ via the action of $\mathrm{Aut}(\mathbb{P}^1)$:

$$\begin{tikzcd} p^{-1}(V) =: U \arrow[r, "i"] \arrow[d, "p"'] & X \\ V & \end{tikzcd}$$

We define $\mathrm{Locus}(V)$ to be the image of U in X; we say that V is a dominating family if $\overline{\mathrm{Locus}(V)} = X$.

Remark 2.2. If V is a dominating family of rational curves, then its general member is a free rational curve. In particular, by [**15**, II.3.7], if B is a subset of X of codimension ≥ 2, a general curve in V does not meet B.

Corollary 2.3. *Let $\sigma \colon X \to Y$ be a smooth blow-up with center B of codimension ≥ 2 and exceptional locus E, let V be a dominating family of rational curves for Y and let V^* be a family of deformations of the strict transform of a general curve in Y. Then $E \cdot V^* = 0$.*

For every point $x \in \mathrm{Locus}(V)$, we will denote by V_x the subscheme of V parametrizing rational curves passing through x.

Definition 2.4. Let V be a family of rational curves on X. We say that

- V is unsplit if it is proper;
- V is locally unsplit if every component of V_x is proper for the general $x \in \mathrm{Locus}(V)$.

Proposition 2.5. [**15**, IV.2.6] *Let X be a smooth projective variety and V an unsplit family of rational curves. Then for every point $x \in \mathrm{Locus}(V)$ we have*

(a) $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
(b) $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

In case V is the unsplit family of deformations of an extremal rational curve of minimal degree, Proposition 2.5 gives the fiber locus inequality:

PROPOSITION 2.6. [**13, 27**] *Let φ be a Fano-Mori contraction of X and E its exceptional locus; let F be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1$$

where $l = \min\{-K_X \cdot C \mid C$ is a rational curve in $F\}$. If φ is the contraction of an extremal ray R, then l is called the length of the ray*.*

DEFINITION 2.7. We define a Chow family of rational curves $\mathcal{V}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles.
If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$ is called the Chow family associated to V. We will usually denote the Chow family associated to a family with the calligraphic version of the same letter.

DEFINITION 2.8. We denote by $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_k$ with the following properties:

- C_i belongs to the family $\mathcal{V}^i$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $x \in C_1 \cup \cdots \cup C_k$,

i.e. $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ is the set of points which belong to a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

DEFINITION 2.9. We denote by $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_k$ with the following properties:

- C_i belongs to the family $\mathcal{V}^i$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_k$,

i.e. $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

DEFINITION 2.10. Let $V^1, \ldots, V^k$ be unsplit families on X. We will say that $V^1, \ldots, V^k$ are numerically independent if their numerical classes $[V^1], \ldots, [V^k]$ are linearly independent in the vector space $N_1(X)$. If moreover $C \subset X$ is a curve we will say that $V^1, \ldots, V^k$ are numerically independent from C if the class of C in $N_1(X)$ is not contained in the vector subspace generated by $[V^1], \ldots, [V^k]$.

LEMMA 2.11. [**2**, Lemma 5.4] *Let $Y \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Y are numerically independent from curves in V, and that $Y \cap \mathrm{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \mathrm{Locus}(V)$*

(a) $\dim \operatorname{Locus}(V)_Y \geq \dim(Y \cap \operatorname{Locus}(V)) + \dim \operatorname{Locus}(V_y)$;
(b) $\dim \operatorname{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$.

Moreover, if $V^1, \ldots, V^k$ are numerically independent unsplit families such that curves contained in G are numerically independent from curves in $V^1, \ldots, V^k$ then either $\operatorname{Locus}(V^1, \ldots, V^k)_Y = \emptyset$ *or*

(c) $\dim \operatorname{Locus}(V^1, \ldots, V^k)_Y \geq \dim Y + \sum(-K_X \cdot V^i) - k$.

Definition 2.12. We define on X a relation of rational connectedness with respect to $\mathcal{V}^1, \ldots, \mathcal{V}^k$ in the following way: x and y are in $\operatorname{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation if there exists a chain of rational curves in $\mathcal{V}^1, \ldots, \mathcal{V}^k$ which joins x and y, i.e. if $y \in \operatorname{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_x$ for some m.

To the $\operatorname{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation we can associate a fibration, at least on an open subset.

Theorem 2.13. [**10**], [**15**, IV.4.16] *There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi \colon X^0 \to Z^0$ such that*

(a) *the $rc(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation restricts to an equivalence relation on X^0;*
(b) *the fibers of π are equivalence classes for the $rc(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation;*
(c) *for every $z \in Z^0$ any two points in $\pi^{-1}(z)$ can be connected by a chain of at most $2^{\dim X - \dim Z} - 1$ cycles in $\mathcal{V}^1, \ldots, \mathcal{V}^k$.*

Definition 2.14. In the above assumptions, if π is the constant map we say that X is $\operatorname{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connected.

Definition 2.15. A minimal horizontal dominating family with respect to π is a family V of horizontal curves such that $\operatorname{Locus}(V)$ dominates Z^0 and $-K_X \cdot V$ is minimal among the families with this property.
If π is the identity map we say that V is a minimal dominating family for X.

Definition 2.16. Let $\mathcal{V}$ be the Chow family associated to a family of rational curves V. We say that V is quasi-unsplit if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V.
We say that V is locally quasi-unsplit if, for a general $x \in \operatorname{Locus}(\mathcal{V})$ every component of any reducible cycle in $\mathcal{V}_x$ is numerically proportional to V

Note that any family of deformations of a rational curve whose numerical class lies in an extremal ray of $\operatorname{NE}(X)$ is quasi-unsplit.

Lemma 2.17. *Let X be a manifold and let L be a line bundle on X Let V be a family of rational curves such that $L \cdot V > 0$. Then there exists an unsplit family*

V^L such that $L \cdot V^L > 0$ and

$$[V] \equiv [V^L] + [\Delta],$$

where Δ is an effective rational one cycle.

Proof. If V is unsplit there is nothing to prove, so assume that the associated Chow family $\mathcal{V}$ contains a reducible cycle $\sum \Gamma_i$: then for at least one i we have $L \cdot \Gamma_i > 0$.
Let V^i be a family of deformations of Γ_i; if V^i is unsplit set $V^L = V^i$, otherwise let $\sum \Gamma_{ij}$ be a reducible cycle in the associated Chow family $\mathcal{V}^i$: then for at least one j we have $L \cdot \Gamma_{ij} > 0$.
Let V^{ij} be a family of deformations of Γ_{ij}; if V^{ij} is unsplit set $V^L = V^{ij}$, otherwise continue as above. Since the degree of V with respect to an ample line bundle is finite the procedure ends after a finite number of steps. $\square$

Notation: Let S be a subset of X. We write $N_1(S) = \langle V^1, \ldots, V^k \rangle$ if the numerical class in $\mathrm{N}_1(X)$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i[C_i]$, with $a_i \in \mathbb{Q}$ and $C_i \in V^i$. We write $\mathrm{NE}(S) = \langle V^1, \ldots, V^k \rangle$ (or $\mathrm{NE}(S) = \langle R_1, \ldots, R_k \rangle$) if the numerical class in $\mathrm{N}_1(X)$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i[C_i]$, with $a_i \in \mathbb{Q}_{\geq 0}$ and $C_i \in V^i$ (or $[C_i]$ in R_i).

Lemma 2.18. [**6**, Lemma 1.4.5], [**19**, Lemma 1], [**11**, Corollary 2.23] *Let $Y \subset X$ be a closed subset and V an unsplit family of rational curves. Then every curve contained in $\mathrm{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients*

$$aC_Y + bC_V,$$

where C_Y is a curve in Y, C_V belongs to the family V and $a \geq 0$.
Moreover, if Σ is an extremal face of $\mathrm{NE}(X)$, Y is a fiber of the associated contraction and $[V]$ does not belong to Σ, then

$$\mathrm{NE}(\mathrm{ChLocus}_m(V)_Y) = \langle \Sigma, [V] \rangle \quad \text{for every} \quad m \geq 1.$$

3. Dominating families and Picard number

We collect in this section some technical result that we will need in the proof. The first is a variation of a classical construction of Mori theory, and says that, given a family of rational curves V and a curve C contained in $\mathrm{Locus}(V_x)$ for an x such that V_x is proper we have $[C] \equiv a[V]$.
The only new remark - which already followed from the old proofs, but, to our best knowledge, was not stated - is the fact that a is a positive integer.

Lemma 3.1. *Let X be a smooth variety, V a family of rational curves on X, $x \in \mathrm{Locus}(V)$ a point such that V_x is proper and C a curve contained in $\mathrm{Locus}(V_x)$. Then C is numerically equivalent to an integral multiple of a curve in V.*

Proof. Consider the basic diagram

$$\begin{tikzcd} p^{-1}(V_x) =: U_x \arrow[r, "i"] \arrow[d, "p"'] & X \\ V_x & \end{tikzcd} \tag{3.1.1}$$

Let C be a curve contained in $\mathrm{Locus}(V_x)$; if C is a curve parametrized by V we have nothing to prove, so we can suppose that this is not the case.
In particular we have that $i^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of p and dominates C via i; let S' be the surface $p^{-1}(p(C'))$, let B' be the curve $p(C') \subset V_x$ and let $\nu\colon B \to B'$ be the normalization of B'. By base change we obtain the following diagram

$$\begin{tikzcd} S_B \arrow[r, "\bar\nu"] \arrow[d] & U_x \arrow[r, "i"] \arrow[d, "p"] & X \\ B \arrow[r, "\nu"'] & V_x & \end{tikzcd}$$

Let now $\mu\colon S \to S_B$ be the normalization of S_B; by standard arguments (see for instance [**26**, 1.14]) it can be shown that S is a ruled surface over the curve B; let $j\colon S \to X$ be the composition of i, $\bar\nu$ and μ. Since every curve parametrized by S passes through x there exists an irreducible curve $C_x \subset S$ which is contracted by j; by [**15**, II.5.3.2] we have $C_x^2 < 0$, hence C_x is the minimal section of S.
Since every curve in S is algebraically equivalent to a linear combination with integral coefficients of C_x and a fiber f, and since C_x is contracted by j, every curve in $j(S)$ is algebraically equivalent in X to an integral multiple of $j_*(f)$, which is a curve of the family V; but algebraic equivalence implies numerical equivalence and so the lemma is proved. □

Corollary 3.2. *Let X be a smooth variety of dimension n and let V be a locally unsplit dominating family such that $-K_X \cdot V = n+1$; then $X \simeq \mathbb{P}^n$.*

Proof. For a general point $x \in X$ we know that V_x is proper and $X = \mathrm{Locus}(V_x)$ by Proposition 2.5 (b). Therefore, by Lemma 3.1, for every curve C in X we have $-K_X \cdot C \geq n+1$ and we can apply [**14**, Theorem 1.1]. □

Remark 3.3. The corollary also followed from the arguments in the proof of [**14**, Theorem 1.1].

In the rest of the section we establish some bounds on the Picard number of Fano manifolds with minimal dominating families of high anticanonical degree.

LEMMA 3.4. *Let X be a Fano manifold of dimension $n \geq 3$ and pseudoindex $i_X \geq 2$ with a minimal dominating family W such that $-K_X \cdot W > 2$; if X contains an effective divisor D such that* $\mathrm{NE}(D) = \langle [W] \rangle$ *then $\rho_X = 1$.*

PROOF. The effective divisor D has positive intersection number with at least one of the extremal rays of X. Let R be such a ray, denote by φ_R the associated contraction and by V^R a family of deformations of a minimal rational curve in R.
If the numerical class of W does not belong to R then D cannot contain curves whose numerical class is in R, therefore every fiber of φ_R is one-dimensional.
By Proposition 2.6 this is possible only if $l(R) \leq 2$ and therefore, since $l(R) \geq i_X$, it must be $l(R) = i_X = 2$.
Since every fiber of φ_R is one-dimensional we have, for every $x \in \mathrm{Locus}(V^R)$ that $\dim \mathrm{Locus}(V^R_x) = 1$ and therefore, by Proposition 2.5 (a) V^R is a dominating family.
But, recalling that

$$2 = -K_X \cdot V^R < -K_X \cdot W,$$

we contradict the assumption that W is minimal.
It follows that $[W] \in R$, so the family W is quasi-unsplit and $D \cdot W > 0$; hence X can be written as $X = \mathrm{Locus}(\mathcal{W})_D$, and by Lemma 2.18 we have $\rho_X = 1$. □

COROLLARY 3.5. *Let X be a Fano manifold of dimension $n \geq 3$ and pseudoindex $i_X \geq 2$ which admits a minimal dominating family W such that $-K_X \cdot W \geq n$; then $\rho_X = 1$.*

PROOF. Let $x \in X$ be a general point; every minimal dominating family is locally unsplit, hence $\mathrm{NE}(\mathrm{Locus}(W_x)) = \langle [W] \rangle$ by Lemma 2.18.
By Proposition 2.5 we have $\dim \mathrm{Locus}(W_x) \geq -K_X \cdot W - 1 \geq n-1$, so either $X = \mathrm{Locus}(W_x)$ or $\mathrm{Locus}(W_x)$ is an effective divisor verifying the assumptions of Lemma 3.4. In both cases we can conclude that $\rho_X = 1$. □

LEMMA 3.6. *Let X be a Fano manifold of dimension $n \geq 3$ and pseudoindex $i_X \geq 2$, with a minimal dominating family W such that $-K_X \cdot W = n-1$; let $U \subset X$ be the open subset of points $x \in X$ such that W_x is unsplit. If a general curve C of W is contained in U then either* $\mathrm{Locus}(W)_C$ *is a divisor and $\rho_X = 1$ or there exists an unsplit family V such that $-K_X \cdot V = 2$, $D := \mathrm{Locus}(V)$ is a divisor and $D \cdot W > 0$.*

Proof. Let C be a general curve in W and consider $\mathrm{Locus}(W)_C$; by our assumptions we have $\mathrm{NE}(\mathrm{Locus}(W)_C) = \langle [W] \rangle$ and $\dim \mathrm{Locus}(W)_C \geq n-2$.
If $X = \mathrm{Locus}(W)_C$ then clearly $\rho_X = 1$, while if $\mathrm{Locus}(W)_C$ has codimension one we conclude by Lemma 3.4.
Therefore we can assume that, for a general C in W, each component of $\mathrm{Locus}(W)_C$ has codimension two in X. The fibration $\pi \colon X \dashrightarrow Z$ associated to the open prerelation defined by W is proper, since a general fiber F coincides with $\mathrm{Locus}(W_x)$ for a general $x \in F$ and $\mathrm{Locus}(W_x)$ is closed since W is locally unsplit.
Being π proper there exists a minimal horizontal dominating family V with respect to π; since the general fiber of π has dimension $n-2$, then $\dim Z = 2$, hence for a general $x \in \mathrm{Locus}(V)$ we have $\dim \mathrm{Locus}(V_x) \leq 2$.
It follows that V is an unsplit family, which cannot be dominating by the minimality of W, so $\dim \mathrm{Locus}(V_x) \geq i_X \geq 2$, and $D = \mathrm{Locus}(V)$ is a divisor by Proposition 2.5. Since D dominates Z we have $D \cdot W > 0$. $\square$

4. Fano manifolds obtained blowing-up non Fano manifolds

We start now the proof of our results. Let us fix once and for all the setup and the notation:

4.1. *X is a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, which has a contraction $\sigma \colon X \to Y$ which is the blow-up of a manifold Y along a smooth subvariety B of dimension i_X. We denote by R_σ the extremal ray corresponding to σ, by l_σ its length and by E its exceptional locus.*

Remark 4.2. The assumption on $\dim B$ is equivalent to

$$l_\sigma + i_X = n - 1.$$

In this section we will deal with Fano manifolds as in Theorem 1.1 which are obtained as a blow-up $\sigma \colon X \to Y$ of a manifold Y which is not Fano. It turns out that there is only one possibility (Corollary 4.4) we start with a slightly general result:

Theorem 4.3. *Let X, R_σ and E be as in 4.1 and assume that there exists on X an unsplit family of rational curves V such that $E \cdot V < 0$; then either $[V] \in R_\sigma$ or $X = \mathbb{P}_{\mathbb{P}^{n-3} \times \mathbb{P}^2}(\mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$.*

Proof. Since $E \cdot V < 0$ then $\mathrm{Locus}(V) \subseteq E$, so V is not a dominating family. Pick $x \in \mathrm{Locus}(V)$ and let F_σ be the fiber of σ through x; we have

$$\dim E \geq \dim \mathrm{Locus}(V_x) + \dim F_\sigma \geq i_X + l_\sigma \geq n - 1,$$

so all the above inequalities are equalities; in particular we have $\dim \mathrm{Locus}(V_x) = i_X$ and so, by Proposition 2.5,

$$\dim \mathrm{Locus}(V) \geq n + i_X - 1 - \dim \mathrm{Locus}(V_x) = n - 1,$$

hence $\mathrm{Locus}(V) = E$; therefore the above (in)equalities are true for every $x \in E$. It follows that σ is equidimensional and so it is a smooth blow-up by [**3**, Theorem 5.1].

Considering V as a family on the smooth variety E we can write

$$n - 1 + i_X = \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq -K_E \cdot V + n - 2,$$

therefore $-K_E \cdot V \leq i_X + 1$; on the other hand

$$-K_E \cdot V = -K_X \cdot V - E \cdot V \geq i_X + 1,$$

forcing $-K_E \cdot V = i_X + 1$ and $E \cdot V = -1$.
Then on E we have two unsplit dominating families of rational curves verifying the assumptions of [**19**, Theorem 1], hence $E \simeq \mathbb{P}^{i_X} \times \mathbb{P}^{l_\sigma}$; in particular $\rho_E = 2$.
Now let R be an extremal ray of X such that $E \cdot R > 0$; by [**18**, Corollary 2.15] the contraction φ_R associated to R is a $\mathbb{P}^1$-bundle; in particular, by Proposition 2.6, this implies that $i_X = 2$.
Moreover, denoted by V^R a family of deformation of a minimal rational curve in R, we have $X = \mathrm{Locus}(V^R)_E$, so $\rho_X = 3$ and the description of X is obtained arguing as in the proof of Proposition 7.3 in [**18**]. $\square$

Corollary 4.4. *In the assumptions of Theorem 1.1 either Y is a Fano manifold or $X = \mathbb{P}_{\mathbb{P}^{n-3} \times \mathbb{P}^2}(\mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$, $Y \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)^{n-2})$ and $B \simeq \mathbb{P}^2$ is the section corresponding to the surjection $\mathcal{O} \oplus \mathcal{O}(1)^{n-2} \to \mathcal{O}$.*

Proof. If Y is not Fano then by [**27**, Proposition 3.4] there exists an extremal ray $R' \in \mathrm{NE}(X)$ such that $E \cdot R' < 0$. $\square$

Remark 4.5. Note that, if $X \simeq \mathbb{P}_{\mathbb{P}^{n-3} \times \mathbb{P}^2}(\mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$, then $\mathrm{NE}(X)$ is generated by three extremal rays: one – the $\mathbb{P}^1$-bundle contraction – is of fiber type, while the other two are smooth blow-ups with the same exceptional locus. In particular $\mathrm{NE}(X)$ is as in Theorem 1.1, case (d).

Corollary 4.6. *Let X, R_σ and E be as in 4.1; assume that Y is a Fano manifold and that there exists on X a family of rational curves V such that $E \cdot V < 0$; then $-K_X \cdot V \geq l_\sigma$; moreover, if V is unsplit then $[V] \in R_\sigma$.*

Proof. Arguing as in Lemma 2.17 we can find an unsplit family V^E such that $E \cdot V^E < 0$ and $[V] \equiv [V^E] + [\Delta]$. By Theorem 4.3 we have that $[V^E] \in R_\sigma$, hence

$$-K_X \cdot V \geq -K_X \cdot V^E \geq l_\sigma.$$

To prove the last assertion note that, if V is an unsplit family, we can apply Theorem 4.3 directly to V. $\square$

5. Manifolds with a dominating (quasi)-unsplit family

In this section we will describe the cone of curves of Fano manifolds as in 4.1 which admit a dominating quasi-unsplit family of rational curves W, and such that the target of the blow-up $\sigma\colon X \to Y$ is a Fano manifold.

If the family W is quasi-unsplit but not unsplit then the result can be obtained easily:

Lemma 5.1. *Assume that W is not unsplit; then $\rho_X = 2$, $i_X = 2$ and* $\mathrm{NE}(X) = \langle R_\sigma, [W] \rangle$.

Proof. Since W is not unsplit we have $-K_X \cdot W \geq 2i_X$. Consider the associated Chow family $\mathcal{W}$ and the rc$\mathcal{W}$-fibration $\pi\colon X \dashrightarrow Z$; since a general fiber of π contains $\mathrm{Locus}(W_x)$ for some x, and by Proposition 2.5 we have $\dim \mathrm{Locus}(W_x) \geq -K_X \cdot W \geq 2i_X - 1$ we have

$$\dim Z \leq n + 1 - 2i_X \leq n - 1 - i_X = \dim F_\sigma,$$

where F_σ is a fiber of σ.
A family V^σ of deformations of a minimal curve in R_σ is thereby horizontal and dominating with respect to π; moreover, since F_σ dominates Z we have that $X = \mathrm{Locus}(\mathcal{W})_{F_\sigma}$, hence $\mathrm{NE}(X) = \langle R_\sigma, [W] \rangle$ by Lemma 2.18. $\square$

In view of Lemma 5.1, we can assume throughout the section that W is an unsplit dominating family.

Lemma 5.2. *Let X be a Fano manifold with $\rho_X = 3$. Assume that there exists an effective divisor E which is negative on one extremal ray R of* $\mathrm{NE}(X)$ *and is nonnegative on the other extremal rays. If $E \cdot C = 0$ for a curve $C \subset X$ whose numerical class lies in* $\partial\,\mathrm{NE}(X)$*, then $[C]$ is contained in a two-dimensional face of* $\mathrm{NE}(X)$ *which contains R.*

Proof. By assumption, neither E nor $-E$ are nef, hence the hyperplane $\{E = 0\}$ has nonempty intersection with the interior of $\mathrm{NE}(X)$. Let Σ be a two-dimensional face of $\mathrm{NE}(X)$ containing $[C]$: by the above discussion E cannot be

trivial on the whole face Σ.
Therefore, if $[C]$ lies in the interior of Σ then E must be negative on one of the rays spanning Σ, hence $R \in \Sigma$. If $[C]$ lies on an extremal ray, then E has different sign on the rays which span with $[C]$ a two-dimensional face of $\mathrm{NE}(X)$, so E is negative on one of them, which has to be R. $\square$

LEMMA 5.3. *Assume that there exists an extremal ray R_τ such that $[W] \notin R_\tau$ and either $E \cdot R_\tau > 0$ or $E \cdot W > 0$. Then every fiber of the contraction τ associated to R_τ has dimension not greater than two. In particular τ is either a fiber type contraction or a smooth blow-up of a codimension three subvariety, and in this case the exceptional locus of τ is $\mathrm{Exc}(\tau) = \mathrm{Locus}(W, V^\tau)_{F_\sigma}$, for some fiber F_σ of σ.*

PROOF. Let F_τ be a fiber of τ. If $E \cdot R_\tau > 0$ there exists a fiber F_σ of σ which meets F_τ; since W is dominating we have $F_\sigma \subset \mathrm{Locus}(W)_{F_\sigma}$ and therefore $F_\tau \cap \mathrm{Locus}(W)_{F_\sigma} \neq \emptyset$.
If else $E \cdot W > 0$ then $E \cap \mathrm{Locus}(W)_{F_\tau} \neq \emptyset$, so there exists a fiber F_σ of σ such that $F_\sigma \cap \mathrm{Locus}(W)_{F_\tau} \neq \emptyset$; equivalently, we have that $F_\tau \cap \mathrm{Locus}(W)_{F_\sigma} \neq \emptyset$.
In both cases, this intersection cannot be of positive dimension, since every curve in F_τ has numerical class belonging to R_τ, while every curve in $\mathrm{Locus}(W)_{F_\sigma}$ has numerical class contained in the cone $\langle R_\sigma, [W] \rangle$. By our assumptions

$$\dim \mathrm{Locus}(W)_{F_\sigma} \geq \dim F_\sigma + i_X - 1 \geq l_\sigma + i_X - 1 \geq n - 2,$$

hence $\dim F_\tau \leq 2$. Proposition 2.6 implies that τ cannot be a small contraction; if it is divisorial, by the same inequality it is equidimensional with two-dimensional fibers, so it is a smooth blow-up by [**3**, Theorem 5.1].
In this last case, denoted by V^τ a family of deformations of a minimal curve in R_τ, we have

$$\dim \mathrm{Locus}(W, V^\tau)_{F_\sigma} \geq n - 1,$$

hence $\mathrm{Exc}(\tau) = \mathrm{Locus}(W, V^\tau)_{F_\sigma}$. $\square$

LEMMA 5.4. *Assume that $E \cdot W = 0$. Let $\pi \colon X \dashrightarrow Z$ be the rcW-fibration and let V be a minimal horizontal dominating family with respect to π. Then R_σ, W and V are numerically independent. In particular $\rho_X \geq 3$.*

PROOF. Since $E \cdot W = 0$, E does not dominate Z, hence E cannot contain $\mathrm{Locus}(V)$ and therefore $E \cdot V \geq 0$.
Let $\mathcal{H}$ be the pull-back to X of a very ample divisor in $\mathrm{Pic}(Z)$; $\mathcal{H}$ is zero on curves in the family W and it is positive outside the indeterminacy locus of π; in particular $\mathcal{H} \cdot V > 0$.
If $[V]$ were contained in the plane spanned by R_σ and $[W]$ we could write $[V] =$

$\alpha[V^\sigma] + \beta[W]$, but intersecting with E we would get $\alpha \leq 0$, while intersecting with $\mathcal{H}$ we would get $\alpha > 0$, a contradiction which proves the lemma. $\square$

PROPOSITION 5.5. *Assume that $E \cdot W = 0$. Let π be the rcW-fibration and let V be a minimal horizontal dominating family with respect to π. Then V is unsplit.*

PROOF. Assume first that $E \cdot V > 0$.
If V is not unsplit we will have, for a general $x \in \text{Locus}(V)$, that

$$\dim \text{Locus}(V_x) \geq 2i_X - 1 \geq 3.$$

Since $E \cdot V > 0$, then $E \cap \text{Locus}(V_x) \neq \emptyset$, therefore $\text{Locus}(V_x)$ meets a fiber F_σ of σ. Moreover, since W is dominating, $F_\sigma \subset \text{Locus}(W)_{F_\sigma}$ and so the intersection $\text{Locus}(V_x) \cap \text{Locus}(W)_{F_\sigma}$ is not empty. This fact, together with

$$\dim \text{Locus}(W)_{F_\sigma} \geq l_\sigma + i_X - 1 \geq n - 2,$$

implies that $\text{Locus}(W)_{F_\sigma}$ contains a curve whose class is proportional to $[V]$, a contradiction by Lemma 5.4, since $\text{NE}(\text{Locus}(W)_{F_\sigma}) = \langle [W], R_\sigma \rangle$.

We will now deal with the harder case $E \cdot V = 0$, assuming by contradiction that V is not unsplit.

We claim that E has non zero intersection number with at least one component of a cycle in the Chow family $\mathcal{V}$. To prove the claim, consider the $\text{rc}(W, \mathcal{V})$-fibration $\pi_{W,\mathcal{V}}$; a general fiber of $\pi_{W,\mathcal{V}}$ contains $\text{Locus}(V, W)_x$ for some x, so it has dimension $\geq 3i_X - 2$.
Since E is not contained in the indeterminacy locus of $\pi_{W,\mathcal{V}}$ - which has codimension at least two in X - it meets some fiber G of $\pi_{W,\mathcal{V}}$ which, by semicontinuity, has dimension $\geq 3i_X - 2$. Therefore there exists a fiber F_σ of σ such that $F_\sigma \cap G \neq \emptyset$. and, for such a fiber we have

$$\dim(F_\sigma \cap G) \geq l_\sigma + 3i_X - 2 - n \geq 2i_X - 3 \geq 1;$$

Let C be a curve in $F_\sigma \cap G$; since $C \subset F_\sigma$ we have $E \cdot C < 0$; on the other hand, since $C \subset G$ the numerical class of C can be written as a linear combination of $[W]$ and of classes of irreducible components of cycles in $\mathcal{V}$. Since $E \cdot W = 0$ we see that E cannot have zero intersection number with all the components of cycles in $\mathcal{V}$ and the claim is proved.

So in $\mathcal{V}$ there exists a reducible cycle $\Gamma = \sum_{i=1}^{k} \Gamma_i$ such that $E \cdot \Gamma_1 < 0$. Applying Lemma 2.17 we find an unsplit family T on which E is negative and such that $[\Gamma_1] = [T] + [\Delta]$, with Δ an effective rational 1-cycle.

Since Y is a Fano manifold, by Corollary 4.6 we have that $[T] \in R_\sigma$ and $-K_X \cdot T \geq l_\sigma$; therefore, for a general $x \in \mathrm{Locus}(V)$

$$\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1 = -K_X \cdot (T + \Delta + \sum_{i=2}^{k} \Gamma_i) - 1 \geq l_\sigma + i_X - 1 \geq n - 2.$$

If $\dim \mathrm{Locus}(V_x) \geq n-1$ then $X = \mathrm{Locus}(W)_{\mathrm{Locus}(V_x)}$ and $\rho_X = 2$ against Lemma 5.4; therefore $\dim \mathrm{Locus}(V_x) = -K_X \cdot V - 1 = n - 2$, hence V is a dominating family by Proposition 2.5, $\Gamma = \Gamma_1 + \Gamma_2$, $\Delta = 0$, $\Gamma_1 \in R_\sigma$ and $-K_X \cdot \Gamma_2 = i_X$.

For a general $x \in \mathrm{Locus}(V)$ we have $\dim \mathrm{Locus}(W)_{\mathrm{Locus}(V_x)} \geq n-1$ by Lemma 2.11; moreover, since $\mathrm{NE}(D) = \langle [W], [V] \rangle$ and $\rho_X \geq 3$ by Lemma 5.4, we cannot have $D = X$, hence D is an effective divisor.
We will now reach a contradiction by showing that D has zero intersection number with every extremal ray of X.
Let $\overline{V}$ be any unsplit family whose numerical class is not contained in the plane spanned by $[W]$ and $[V]$; we cannot have $\dim \mathrm{Locus}(\overline{V}_x) = 1$, otherwise $\overline{V}$ would be dominating of anticanonical degree 2, against the minimality of V. This implies that $D \cdot \overline{V} = 0$ since $\mathrm{NE}(D) = \langle [W], [V] \rangle$ implies that $D \cap \mathrm{Locus}(\overline{V}_x) = \emptyset$.
It follows that $D \cdot \Gamma_2 = 0$ and that D is trivial on every extremal ray not lying in the plane $\langle [V], [W] \rangle$. Since $[V] = [\Gamma_1] + [\Gamma_2]$ and $\Gamma_1 \in R_\sigma$, which is a ray not contained in the plane spanned by $[W]$ and $[V]$ we have that also $D \cdot V = 0$.
To conclude it is now enough to observe that we must have $D \cdot W = 0$, otherwise $\mathrm{ChLocus}_2(W)_{\mathrm{Locus}(V_x)} = X$, forcing again $\rho_X = 2$. We have thus reached a contradiciton, since the effective divisor D has to be trivial on the whole $\mathrm{NE}(X)$. $\square$

PROPOSITION 5.6. *Up to replace W with another dominating unsplit family, we can assume that $E \cdot W > 0$.*

PROOF. Assume that $E \cdot W = 0$, let π be the rc$W$-fibration, and let V be a minimal horizontal dominating family with respect to π. By Proposition 5.5 we know that V is unsplit.

Case a) V is dominating.

If $E \cdot V > 0$ the Proposition is proved, so we can assume that $E \cdot V = 0$.
If F_σ is any fiber of σ we have

$$\dim \mathrm{Locus}(V, W)_{F_\sigma} \geq \dim F_\sigma + 2i_X - 2 = l_\sigma + 2i_X - 2 \geq n - 1.$$

Note that, by the assumptions on the intersection numbers, we have $\mathrm{Locus}(V, W)_{F_\sigma} \subseteq E$, and therefore $\mathrm{Locus}(V, W)_{F_\sigma} = E$; in particular it follows from the above inequalities that $i_X = 2$.

We can repeat the same arguments to show that also $\mathrm{Locus}(W,V)_{F_\sigma} = E$; hence every curve contained in E is numerically equivalent to a linear combination

$$a[V^\sigma] + b[V] + c[W]$$

with a, b, $c \geq 0$ by Lemma 2.18, and therefore $\mathrm{NE}(E) = \langle R_\sigma, [V], [W] \rangle$. In particular E has nonpositive intersection with every curve it contains.
Let R_ϑ be an extremal ray such that $E \cdot R_\vartheta > 0$; by [**18**, Corollary 2.15] the associated contraction $\vartheta \colon X \to Y$ is a $\mathbb{P}^1$-bundle; the associated family V^ϑ is dominating and unsplit and $E \cdot V^\vartheta > 0$, and the proposition is proved.

Case b) V is not dominating.

Consider the $\mathrm{rc}(W,V)$-fibration $\pi' \colon X \dashrightarrow Z'$; Z' has positive dimension since by Lemma 5.4 we have $\rho_X \geq 3$.
A general fiber F' of π' contains $\mathrm{Locus}(V,W)_x$ for some $x \in \mathrm{Locus}(V)$, hence $\dim F' \geq 2i_X - 1$ and thus

$$\dim Z' \leq n + 1 - 2i_X \leq l_\sigma.$$

A general fiber F_σ of σ is not contained in the indeterminacy locus of π' and is not contracted by π', since, by Lemma 5.4, $[V]$, $[W]$ and R_σ are numerically independent. Hence we have $\dim Z' \geq \dim F_\sigma = l_\sigma$ and the above inequalities are equalities.
It follows that $i_X = 2$, $\dim Z' = l_\sigma$ and F_σ dominates Z'; this implies that $X = \mathrm{ChLocus}_m(W,V)_{F_\sigma}$ for some m. Therefore $\rho_X = 3$ and so, by Lemma 2.18, the numerical class of every curve in X can be written as

$$\alpha[V^\sigma] + \beta[W] + \gamma[V],$$

with $\alpha \geq 0$. This implies that the plane $\langle [V], [W] \rangle$ is extremal in $\mathrm{NE}(X)$.

The divisor E has to be positive on V, otherwise it would be nonpositive on the whole $\mathrm{NE}(X)$; since $E \cdot W = 0$ then $[W]$ is in an extremal face with R_σ by Lemma 5.2. Since $[W]$ is also in an extremal face with $[V]$ it follows that $[W]$ spans an extremal ray of $\mathrm{NE}(X)$, whose associated contraction is of fiber type.

Let W_Y be a minimal dominating family on Y and let W^* be a family of deformations of the strict transform of a general curve in W_Y.
We have $-K_X \cdot W^* = -K_Y \cdot W_Y \leq n$; in fact, if $-K_X \cdot W^* = n+1$, we would have $Y \simeq \mathbb{P}^n$ by Corollary 3.2 and so $\rho_X = 2$.
Assume that W^* is not locally unsplit; then there exists a reducible cycle $\Gamma = \sum \Gamma_i$ in $\mathcal{W}^*$ such that the family T of deformation of one irreducible component, say Γ_1

is dominating.

We cannot have $E \cdot T = 0$, otherwise, denoting by T_* a family of deformation of the image in Y of a general curve in T we would have

$$-K_X \cdot T = -K_Y \cdot T_* < -K_Y \cdot W_Y$$

against the minimality of W_Y.

Therefore $E \cdot T > 0$; in this case E must be negative on another component of Γ, say Γ_2. By Corollary 4.6, we have that $-K_X \cdot \Gamma_2 \geq l_\sigma$, and thus $\Gamma = \Gamma_1 + \Gamma_2$ and so $-K_X \cdot \Gamma_1 < 2i_X$ and the family T is unsplit and dominating.

We are left with the case in which W^* is locally unsplit.

Consider $\mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W^*_x)}$ for a general $x \in \mathrm{Locus}(W^*)$: it is contained in E and, by Lemma 2.11

$$\dim(\mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W^*_x)}) \geq n - 2 + l_\sigma - 1,$$

yielding $l_\sigma \leq 2$ and so $n = 5$, a contradiction which concludes the proof. □

THEOREM 5.7. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, with a contraction $\sigma \colon X \to Y$ which is the blow-up of a Fano manifold Y along a smooth subvariety B of dimension i_X. If X admits a dominating unsplit family of rational curves W then the possible cones of curves of X are listed in the following table, where R_σ is the ray corresponding to σ, F stands for a fiber type contraction and D_{n-3} for a divisorial contraction whose exceptional locus is mapped to a subvariety of codimension three.*

ρ_X	i_X	R_1	R_2	R_3	R_4
2		R_σ	F		
3	2,3	R_σ	F	F	
3	2	R_σ	F	D_{n-3}	
4	2	R_σ	F	F	F

In particular generalized Mukai conjecture holds for X.

PROOF. Let V^σ be a family of deformations of a minimal rational curve in R_σ. By Proposition 5.6 we can assume that $E \cdot W > 0$; therefore the family V^σ is horizontal and dominating with respect to the rc$W$-fibration $\pi \colon X \dashrightarrow Z$.

It follows that a general fiber F' of the the rc(W, V^σ)-fibration $\pi' \colon X \dashrightarrow Z'$ contains $\mathrm{Locus}(W)_{F_\sigma}$ for some fiber F_σ of σ, and therefore

$$\dim F' \geq \dim \mathrm{Locus}(W)_{F_\sigma} \geq l_\sigma + i_X - 1 \geq n - 2,$$

hence $\dim Z' \leq 2$.

If $\dim Z' = 0$ then X is $\mathrm{rc}(W, V^\sigma)$-connected and $\rho_X = 2$; denote by R_ϑ the extremal ray of $\mathrm{NE}(X)$ different from R_σ. We claim that in this case $[W] \in R_\vartheta$. In fact, if this were not the case, R_ϑ would be a small ray by [**11**, Lemma 2.4], but in our assumptions we have $E \cdot R_\vartheta > 0$, against Lemma 5.3.
We can thus conclude that in this case $\mathrm{NE}(X) = \langle R_\sigma, R_\vartheta\rangle$ and that R_ϑ is of fiber type.

If $\dim Z' > 0$ take V' to be a minimal horizontal dominating family for π'; by [**2**, Lemma 6.5] we have $\dim \mathrm{Locus}(V'_x) \leq 2$, and therefore

$$-K_X \cdot V' \leq \dim \mathrm{Locus}(V'_x) + 1 \leq 3,$$

so V' is unsplit and $i_X \leq 3$.

The classes $[V^\sigma]$ and $[W]$ lie on an extremal face $\Sigma = \langle R_\sigma, R\rangle$ of $\mathrm{NE}(X)$, since, otherwise, by [**11**, Lemma 2.4], X would have a small contraction, against Lemma 5.3. Let $\mathcal{H}$ the pull back via π of a very ample divisor on Z.
We know that $\mathcal{H} \cdot W = 0$ and $\mathcal{H} \cdot R_\sigma > 0$, since V^σ is horizontal and dominating with respect to π. It follows that $[W] \in R$ (and so R is of fiber type), since otherwise the exceptional locus of R would be contained in the indeterminacy locus of π, and thus the associated contraction would be small, contradicting again Lemma 5.3.

Consider now the $\mathrm{rc}(W, V^\sigma, V')$-fibration $\pi'' \colon X \dashrightarrow Z''$: its fibers have dimension $\geq n-1$ and so $\dim Z'' \leq 1$.

If $\dim Z'' = 0$ we have that X is $\mathrm{rc}(W, V^\sigma, V')$-connected and $\rho_X = 3$; by Lemma 5.3 every extremal ray of X has an associated contrsction which is either of fiber type or divisorial.

Assume that there exists an extremal ray R' not belonging to σ such that its associated contraction is of fiber type.
This ray must lie in a face of $\mathrm{NE}(X)$ with R by [**11**, Lemma 5.4].
If $E \cdot R' > 0$ we can exchange the role of R and R' and repeat the previous argument, therefore R' lies in a face with R_σ and $\mathrm{NE}(X) = \langle R_\sigma, R, R'\rangle$.
If $E \cdot R' = 0$ there cannot be any extremal ray in the half-space of $\mathrm{NE}(X)$ determined by the plane $\langle R', R_\sigma\rangle$ and not containing R, otherwise this ray would have negative intersection with E, a contradiction. So again $\mathrm{NE}(X) = \langle R_\sigma, R, R'\rangle$.

So we can assume that every ray not belonging to Σ is divisorial. Let R' be such a ray, denote by E' its exceptional locus, and by W' a family of deformations of a minimal rational curve in R'.
Let F' be a fiber of the $\mathrm{rc}(W, V^\sigma)$-fibration π'; since $\dim F' \geq n-2$ we can write $E' = \mathrm{Locus}(W')_{F'}$. It follows that $\mathrm{NE}(E') = \langle R_\sigma, R, R'\rangle$. In particular E' cannot

be trivial on Σ, otherwise it would be nonpositive on the whole $\mathrm{NE}(X)$.
We claim that R and R' lie on an extremal face of $\mathrm{NE}(X)$: if $E' \cdot R > 0$ the family W' is horizontal and dominating with respect to π and so R' and R are in a face by [**11**, Lemma 5.4]. If else $E' \cdot R = 0$ we have $E' \cdot R_\sigma > 0$. It follows that in the half-space determined by $\langle R, R' \rangle$ and not containing R_σ the divisor E' is negative. Therefore this half space cannot contain an extremal ray R'', since otherwise, the exceptional locus of this ray must be contained in E', contradicting the fact that $\mathrm{NE}(E') = \langle R_\sigma, R, R' \rangle$.
So we have proved that every ray not belonging to Σ lies in a face with R, and this implies that such a ray is unique and $\mathrm{NE}(X) = \langle R_\sigma, R, R' \rangle$.

Recalling that $E' = \mathrm{Locus}(W')_{F'}$ and that $\dim F' \geq n-2$ we have that every fiber of the contraction φ' associated to R' has dimension two; it follows that $i_X = 2$ and that φ' is a smooth blow-up of a codimension three subvariety by [**3**, Theorem 5.1].

If $\dim Z'' = 1$ consider a minimal horizontal dominating family V'' for π'': in this case $\rho_X = 4$, $i_X = 2$ and both V' and V'' are dominating. Let F_σ be a fiber of σ: then we can write $X = \mathrm{Locus}(V', V'')_{\mathrm{Locus}(W)_{F_\sigma}}$. By Lemma 2.18 every curve in X can be written with positive coefficients with respect to V^σ and W; but W, V' and V'' play a symmetric role, so we can conclude that $\mathrm{NE}(X) = \langle R_\sigma, [W], [V'], [V''] \rangle$, and all the three rays different from R_σ are of fiber type. $\square$

6. Manifolds without a dominating quasi-unsplit family

In this section we will show that the only Fano manifold as in 4.1 which does not admit a dominating quasi-unsplit family of rational curves is the blow-up of $\mathbb{G}(1,4)$ along a plane of bidegree $(0,1)$ (Theorem 6.7). In view of Theorem 5.7 this will conclude the proof of Theorem 1.1 and prove Theorem 1.2.

From now on we will thus work in the following setup:

6.1. *X is a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, which does not admit a quasi-unsplit dominating family of rational curves and has a contraction $\sigma\colon X \to Y$ which is the blow-up of a manifold Y along a smooth subvariety B of dimension i_X. We denote by R_σ the extremal ray corresponding to σ, by l_σ its length and by E its exceptional locus.*

In view of Corollary 4.4 we can assume that Y is a Fano manifold. We need some preliminary work to establish some properties of families of rational curves on X and Y.

Lemma 6.2. *Assume that $\rho_X = 2$. Let W' be a minimal dominating family of rational curves for Y and let W^* be a family of deformations of the strict transform of a general curve in W'. Then $-K_Y \cdot W' \geq n-1$.*

Proof. The family W^* is dominating and therefore, by 6.1, not quasi unsplit. Moreover, by Corollary 2.3 we have $E \cdot W^* = 0$, hence there exists a component Γ_1^* of a reducible cycle Γ^* in $\mathcal{W}^*$ such that $E \cdot \Gamma_1^* < 0$.
By Corollary 4.6 we have $-K_X \cdot \Gamma_1^* \geq l_\sigma$, and therefore

$$-K_Y \cdot W' = -K_X \cdot W^* \geq l_\sigma + i_X = n - 1.$$

□

Proposition 6.3. *Let X, Y, R_σ and E be as in 6.1. Then there does not exist on X any locally unsplit dominating family W such that $E \cdot W > 0$.*

Proof. Assume that such a family W exists; we will derive a contradiction showing that in this case $n = 5$.

First of all we prove that $i_X = 2$ and that X is rationally connected with respect to the Chow family $\mathcal{W}$ associated to W and to V^σ, the family of deformations of a general curve of minimal degree in R_σ.
Since $E \cdot W > 0$, for a general $x \in X$, the intersection $E \cap \mathrm{Locus}(W_x)$ is nonempty. On the other hand, the fact that $E \cdot V^\sigma < 0$ yields that the families W and V^σ are numerically independent, and therefore, for every fiber F_σ of σ, we have $\dim(\mathrm{Locus}(W_x) \cap F_\sigma) \leq 0$.
Now, if we denote by F_σ a fiber of σ which meets $\mathrm{Locus}(W_x)$, it follows that

$$2i_X - 1 \leq -K_X \cdot W - 1 \leq \dim \mathrm{Locus}(W_x) \leq n - \dim F_\sigma \leq n - l_\sigma = i_X + 1,$$

whence $i_X = 2$, $\dim \mathrm{Locus}(W_x) = i_X + 1 = 3$ and $-K_X \cdot W = 2i_X = 4$.
In particular $\dim(E \cap \mathrm{Locus}(W_x)) = 2 = \dim B$, hence $\sigma(E \cap \mathrm{Locus}(W_x)) = B$ and every fiber of σ meets $\mathrm{Locus}(W_x)$.
Let x and y be two general points in X; every fiber of σ meets both $\mathrm{Locus}(W_x)$ and $\mathrm{Locus}(W_y)$, so the points x and y can be connected using two curves in W and a curve in V^σ. This implies that X is $\mathrm{rc}(\mathcal{W}, V^\sigma)$-connected.

Our next step consists in proving that $\rho_X = 2$, showing that the numerical class of every irreducible component of any cycle in $\mathcal{W}$ lies in the plane Π spanned in $\mathrm{N}_1(X)$ by $[W]$ and R_σ.
Let $x \in X$ be a general point; by Lemma 2.11 we have

$$\dim \mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W_x)} \geq l_\sigma + 2i_X - 2 \geq n - 1,$$

therefore $E = \mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W_x)}$ and $\mathrm{N}_1(E) = \Pi$ by Lemma 2.18.
We have already proved that $-K_X \cdot W = 4$ and $i_X = 2$; therefore every reducible cycle in $\mathcal{W}$ has exactly two irreducible components, and the families of deformations of these components are unsplit.
Let $\Gamma_1 + \Gamma_2$ be a reducible cycle in $\mathcal{W}$; without loss of generality we can assume that $E \cdot \Gamma_1 > 0$. Denote by W^1 a family of deformations of Γ_1; being unsplit, the family W^1 cannot be dominating, hence for every $x \in \mathrm{Locus}(W^1)$ we have $\dim \mathrm{Locus}(W^1_x) \geq 2$ by Proposition 2.5. Since $E \cap \mathrm{Locus}(W^1_x) \neq \emptyset$ it follows that $\dim(E \cap \mathrm{Locus}(W^1_x)) \geq 1$ for every $x \in \mathrm{Locus}(W^1)$, so $[W^1] \in \Pi$, and consequently also $[W^2] \in \Pi$; it follows that $\rho_X = 2$.

Let now T_Y be a minimal dominating family for Y and let T be the family of deformations of the strict transform of a general curve in T_Y. By Lemma 6.2 we have $-K_X \cdot T = -K_Y \cdot T_Y \geq n - 1$.
By this last inequality, the intersection $\mathrm{Locus}(W_x) \cap \mathrm{Locus}(T_x)$ for a general $x \in X$ has positive dimension; since T is independent from W – recall that $E \cdot T = 0$ and $E \cdot W > 0$ – the family T cannot be locally quasi-unsplit.
Therefore, in the associated Chow family $\mathcal{T}$, there exists a reducible cycle $\Lambda = \Lambda_1 + \Lambda_2$ such that a family of deformations T^1 of Λ_1 is dominating and independent from T.
The family T^1, being dominating, cannot be unsplit, hence $-K_X \cdot T^1 \geq 4$; moreover, since T^1 is also independent from T we have $E \cdot T^1 > 0$. It follows that $E \cdot \Lambda_2 < 0$ and so $-K_X \cdot \Lambda_2 \geq l_\sigma$ by Lemma 4.6. Therefore

$$-K_Y \cdot T_Y = -K_X \cdot T \geq l_\sigma + 2i_X = n + 1$$

so $Y \simeq \mathbb{P}^n$ by Corollary 3.2.
The center B of σ cannot be a linear subspace of Y, since otherwise $i_X + l_\sigma = n+1$; take l to be a proper bisecant of B and let $\widetilde{l}$ be its strict transform: we have

$$2 = i_X \leq -K_X \cdot \widetilde{l} = n + 1 - 2l_\sigma = 4 - l_\sigma,$$

hence $l_\sigma = 2$ and $n = 5$. $\square$

Corollary 6.4. *Let X, Y, R_σ and E be as in 6.1. Then there does not exist any family of rational curves V independent from R_σ such that V_x is unsplit for some $x \in E$ and such that $E \subseteq \overline{\mathrm{Locus}(V)}$.*

Proof. Assume by contradiction that such a family exists.
First of all we prove that V cannot be unsplit. If this is the case, since on X there are no unsplit dominating families it must be $\overline{\mathrm{Locus}(V)} = \mathrm{Locus}(V) = E$.
We can thus apply Lemma 2.11 a) to get that $\dim \mathrm{Locus}(V)_{F_\sigma} = n - 1$ for every

fiber F_σ of σ. It follows that $E = \mathrm{Locus}(V)_{F_\sigma}$ and therefore $\mathrm{NE}(E) = \langle R_\sigma, [V]\rangle$ by Lemma 2.18.
Since V is a dominating unsplit family for the smooth variety E we have $-K_E \cdot V = \dim \mathrm{Locus}(V_x) + 1$, hence, by adjunction, $E \cdot V < 0$; since V is independent from R_σ it follows from Theorem 4.3 that Y is not a Fano manifold, a contradiction.

Since V is not unsplit we have $-K_X \cdot V \geq 2i_X$ and therefore, for a point $x \in E$ such that V_x is unsplit, we have

$$\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1 \geq 2i_X - 1.$$

On the other hand, since V is independent from R_σ, we have, for any fiber F_σ of σ, that $\dim \mathrm{Locus}(V_x) \cap F_\sigma \leq 0$, hence $\dim \mathrm{Locus}(V_x) \leq n - l_\sigma = i_X + 1$.
It follows that $i_X = 2$, $-K_X \cdot V = 4$ and $\dim \mathrm{Locus}(V_x) = 3$; the last two equations, by Proposition 2.5 imply that V is dominating. Moreover, since $-K_X \cdot V = 4$, the family V is also locally unsplit, otherwise we would have a dominating family of lower degree, hence unsplit.
Since $E \cap \mathrm{Locus}(V_x)$ is not empty and we cannot have $\mathrm{Locus}(V_x) \subset E$ – recall that V_x is unsplit and V is independent from R_σ, so $\mathrm{Locus}(V_x)$ can meet fibers of σ only in points – it follows that $E \cdot V > 0$ and we can apply Proposition 6.3. □

Remark 6.5. If $C_Y \subset Y$ is a curve which meets the center B of the blow-up in k points and is not contained in it, then $-K_Y \cdot C_Y \geq n - 1 + (k-1)l_\sigma$.

Proof. Let C be the strict transform of C_Y: then the statement follows from the canonical bundle formula

$$-K_X = -\sigma^* K_Y - l_\sigma E,$$

which yields

$$-K_Y \cdot C_Y = -K_X \cdot C + l_\sigma E \cdot C \geq i_X + k l_\sigma \geq n - 1 + (k-1)l_\sigma.$$

□

Corollary 6.6. *Let W_Y be a minimal dominating family for Y and assume that $-K_Y \cdot W_Y = n - 1$. Assume that there exists a reducible cycle Γ in $\mathcal{W}_Y$ which meets B; then $\Gamma \subset B$ and $\mathrm{NE}(B) = \langle [W_Y] \rangle$.*

Proof. Let Γ_i be a component of Γ: we know that $-K_Y \cdot \Gamma_i < n - 1$, so the whole cycle Γ has to be contained in B by remark 6.5.
Let W_Y^i be a family of deformations of Γ_i; the pointed locus $\mathrm{Locus}(W_Y^i)_b$ is contained in B for every $b \in B$, again by remark 6.5, hence

$$-K_Y \cdot W_Y^i \leq \dim \mathrm{Locus}(W_Y^i)_b \leq \dim B = i_X \leq i_Y,$$

where the last inequality follows from [**7**, Theorem 1, (iii)].
Therefore T^i is unsplit and $B = \mathrm{Locus}(W_Y^i)_b$, hence $\mathrm{NE}(B) = \langle [W_Y^i] \rangle$. It follows that all the components Γ_i of Γ are numerically proportional, and thus they are all numerically proportional to W_Y. □

We are now ready to prove the following

THEOREM 6.7. *Let X be a Fano manifold of dimension $n \geq 6$ and pseudoindex $i_X \geq 2$, which is the blow-up of another Fano manifold Y along a smooth subvariety B of dimension i_X; assume that X does not admit a quasi-unsplit dominating family of rational curves. Then $Y \simeq \mathbb{G}(1,4)$ and B is a plane of bidegree $(0,1)$.*

PROOF. The proof is quite long and complicated; we will divide it into different steps, in order to make our procedure clearer.

Step 1 *A minimal dominating family of rational curves on Y has anticanonical degree $n-1$.*

Let W_Y be a minimal dominating family of rational curves for Y, and let W be the family of deformations of the strict transform of a general curve in W_Y.
Apply [**4**, Lemma 4.1] to W (note that in the proof of that lemma the minimality of W is not needed). The first case in the lemma cannot occur by Corollary 6.4, so there exists a reducible cycle $\Gamma = \Gamma_\sigma + \Gamma_V + \Delta$ in $\mathcal{W}$ with $[\Gamma_\sigma]$ belonging to R_σ, Γ_V belonging to a family V, independent from R_σ, such that V_x is unsplit for some $x \in E$, and Δ an effective rational 1-cycle. In particular

$$(6.7.2) \qquad -K_X \cdot W \geq -K_X \cdot (\Gamma_\sigma + \Gamma_V + \Delta) \geq l_\sigma + i_X \geq n-1.$$

By the canonical bundle formula and Corollary 2.3 we have that

$$-K_Y \cdot W_Y = -K_X \cdot W \geq n-1.$$

If $-K_Y \cdot W_Y = n+1$ then Y is a projective space by Corollary 3.2. The center of σ cannot be a linear subspace, otherwise X would admit an unsplit dominating family of rational curves; then if l is a proper bisecant of B and $\widetilde{l}$ is its strict transform we have

$$2 \leq i_X \leq -K_X \cdot \widetilde{l} = n+1-2l_\sigma = 4 - l_\sigma,$$

hence $l_\sigma = 2$ and $n = 5$, against the assumptions.

We can thus assume that $-K_Y \cdot W_Y \leq n$.
Note that, by (6.7.2), the reducible cycle Γ has only two irreducible components Γ_σ and Γ_V; moreover the class of Γ_σ is minimal in R_σ, hence $E \cdot \Gamma_\sigma = -1$, and $-K_X \cdot V \leq i_X + 1$. In particular V is an unsplit family.
Recalling that $E \cdot W = 0$ we get $E \cdot \Gamma_V = 1$. Geometrically, a general curve in V

is the strict transform of a curve in W_Y which meets B in one point; moreover, since a curve in W_Y not contained in B cannot meet B in more than one point by Remark 6.5, we have that

$$\sigma(\mathrm{Locus}(V) \setminus E) = \mathrm{Locus}(W_Y)_B \setminus B. \tag{6.7.3}$$

Assume that $-K_Y \cdot W_Y = n$; in this case $\rho_Y = 1$ by Corollary 3.5.
For a general point $y \in Y$, we have that $\mathrm{Locus}(W_Y)_y$ is an effective, hence ample, divisor, so it meets B. In particular we have $\dim \mathrm{Locus}(W_Y)_B = n$, and by (6.7.3) this implies that V is dominating, against the assumptions since V is unsplit. This completes step 1.

Step 2 *The strict transforms of curves in a minimal dominating family on Y which meet B fill up a divisor on X.*

Let x be a point in $E \cap \mathrm{Locus}(V)$ and let F_σ be the fiber of σ containing x; since $\dim F_\sigma + \dim \mathrm{Locus}(V_x) \leq n$ we have

$$\dim \mathrm{Locus}(V_x) \leq n - l_\sigma = i_X + 1.$$

By inequality 2.5 we have that $\dim \mathrm{Locus}(V) \geq n-2$; since V is an unsplit family it cannot be dominating, so we need to show that $\dim \mathrm{Locus}(V) \neq n-2$.
Assume by contradiction that $\dim \mathrm{Locus}(V) = n-2$; in this case, again by inequality 2.5, for every $x \in \mathrm{Locus}(V)$ we have $\dim \mathrm{Locus}(V_x) = i_X + 1$, so for every $x \in X$ the intersection $\mathrm{Locus}(V_x) \cap E$ dominates B.
Consider a point $x \in \mathrm{Locus}(V) \setminus E$, denote by y its image $\sigma(x)$ and consider $\mathrm{Locus}(\mathcal{W}_Y)_y$: since $\mathrm{Locus}(V_x) \cap E$ dominates B, we have $B \subset \mathrm{Locus}(\mathcal{W}_Y)_y$. But cycles in $\mathcal{W}_Y$ passing through y and meeting B are irreducible by corollary 6.6, so $B \subseteq \mathrm{Locus}(W_Y)_y$ and by Lemma 3.1 the numerical class of every curve in B is an integral multiple of $[W_Y]$. This fact together with Corollary 6.6 allows us to conclude that B does not meet any reducible cycle in $\mathcal{W}_Y$.

We claim that a general curve C of W_Y is contained in the open subset U of points $y \in Y$ such that $(W_Y)_y$ is proper. If this were not true, then $\mathrm{Locus}(W_Y) \setminus U$ should have codimension one, and so there would exist a family W_Y^1 of deformations of an irreducible component of a cycle in $\mathcal{W}_Y$ whose locus is a divisor; moreover this divisor should have positive intersection number with W_Y.
This last condition would imply that $\mathrm{Locus}(W_Y^1)$ has nonempty intersection with B, since the numerical class of any curve in B is an integral multiple of $[W_Y]$, but we have proved that B does not meet any reducible cycle in $\mathcal{W}_Y$, so we have reached a contradiction that proves the claim.

Therefore we can apply Lemma 3.6 and get that $D := \mathrm{Locus}(W_Y)_C$ is a divisor and $\rho_Y = 1$, since in the other case of the lemma we would find a family of anti-canonical degree two meeting B, against Remark 6.5.
Being $\rho_Y = 1$ the effective divisor D is ample, hence it meets B; therefore for a general curve C in W_Y there exists another curve in W_Y which meets both B and C; in other words, a general curve in W_Y meets $\mathrm{Locus}(W_Y)_B$, a contradiction since $\mathrm{Locus}(W_Y)_B$ has codimension two in Y by (6.7.3).

Step 3 *The Picard number of Y is one.*

By (6.7.3) we have that $\dim \mathrm{Locus}(W_Y)_B = \dim \mathrm{Locus}(V) = n-1$. This implies that B contains curves whose numerical class is proportional to $[W_Y]$, otherwise by Lemma 2.11 we would have $\dim \mathrm{Locus}(W_Y)_B = n$.

If B does not meet any reducible cycle in $\mathcal{W}_Y$ we can argue as in the claim in step 2 and conclude that $\rho_Y = 1$.

If else B meets a reducible cycle in $\mathcal{W}_Y$ then, by Corollary 6.6, every curve in B is numerically proportional to $[W_Y]$, hence $\mathrm{NE}(\mathrm{Locus}(W_Y)_B) = \langle [W_Y] \rangle$ and we conclude that $\rho_Y = 1$ by Lemma 3.4.

Step 4 *The families of deformations of the strict transforms of curves in a minimal dominating family on Y which meet B are extremal in* $\mathrm{NE}(X)$.

Let $D = \mathrm{Locus}(V)$; we have $D \cdot W > 0$, since $E \cdot W = 0$ and $\mathrm{Pic}(X) = \langle E, D \rangle$. Therefore $\mathrm{Locus}(W, V)_x = \mathrm{Locus}(V)_{\mathrm{Locus}(W_x)}$ is nonempty for a general $x \in X$, and so has dimension $\geq n - 2 + i_X - 1 \geq n - 1$ by Lemma 2.11. It follows that $i_X = 2$ and $D = \mathrm{Locus}(W, V)_x$.
The last equality, by Lemma 2.18, yields that every curve in D is numerically equivalent to a linear combination $a[W] + b[V]$ with $a \geq 0$.
This implies that $\mathrm{NE}(D)$ is contained in the cone spanned by $[V]$ and by an extremal ray R of $\mathrm{NE}(X)$. Since $E \cdot W = 0$ and $E \cdot V > 0$ it must be $E \cdot R < 0$, so $R = R_\sigma$ and $\mathrm{NE}(D) = \langle R_\sigma, [V] \rangle$.
Let R_τ be the extremal ray of $\mathrm{NE}(X)$ different from R_σ and denote by τ the associated contraction. The contraction τ is birational, since X does not admit quasi-unsplit dominating families of rational curves, therefore its fibers have dimension at least two by inequality 2.6.
We claim that $[V] \in R_\tau$; if we assume that this is not the case then $D \cap \mathrm{Exc}(\tau) = \emptyset$, since $\mathrm{NE}(D) = \langle R_\sigma, [V] \rangle$. In particular $D \cdot R_\tau = 0$, so $D \cdot R_\sigma > 0$ by [**8**, Lemma 2.1]. By the same lemma the effective divisor E is positive on R_τ.
Let F_σ and F_τ be two meeting fibers of the contractions σ and τ respectively; we

have $\dim(F_\sigma \cap F_\tau) = 0$, hence

$$n \geq \dim F_\sigma + \dim F_\tau \geq l_\sigma + l_\tau.$$

Therefore, recalling that $i_X = 2$ and thus $l_\sigma = n - 3$, we have $l_\tau \leq 3$, so $\dim \mathrm{Exc}(R_\tau) \geq n - 2$ by inequality 2.6.
In particular, if F_σ is a fiber of σ meeting $\mathrm{Exc}(\tau)$ we have

$$\dim(F_\sigma \cap \mathrm{Exc}(\tau)) \geq l_\sigma - 2 \geq 1.$$

Let C be a curve in $F_\sigma \cap \mathrm{Exc}(\tau)$; since $D \cdot R_\sigma > 0$ we have $D \cap C \neq \emptyset$, hence $D \cap \mathrm{Exc}(\tau) \neq \emptyset$, a contradiction that proves the extremality of $[V]$.

Step 5 *The contraction of X different from σ is the blow-up of $\mathbb{P}^n$ along a smooth subvariety of codimension three.*

Since $\mathrm{Exc}(\tau) = D = \mathrm{Locus}(W, V)_x$ every fiber of τ is two-dimensional; we can apply [**3**, Theorem 5.1] to get that $\tau\colon X \to Z$ is a smooth blow-up.
Let T_Z be a minimal dominating family for Z and T^* a family of deformations of the strict transform of a general curve in T_Z.
Among the families of deformations of the irreducible components of cycles in $\mathcal{T}^*$ there is at least one family which is dominating and locally unsplit; call it T.
Being dominating, T cannot be quasi-unsplit, so we have $-K_X \cdot T \geq 4$, hence for a general $x \in X$ we have $\dim \mathrm{Locus}(T_x) \geq 3$. Since T is locally unsplit we also have $\mathrm{NE}(\mathrm{Locus}(T_x)) = \langle [T] \rangle$.
On the other hand $\dim \mathrm{Locus}(W_x) \geq n - 2$, so $\dim(\mathrm{Locus}(T_x) \cap \mathrm{Locus}(W_x)) \geq 1$. Therefore T is numerically proportional to W, since $\mathrm{NE}(\mathrm{Locus}(W_x)) = \langle [W] \rangle$.
If $-K_X \cdot T < -K_X \cdot W$ then the images in Y of the curves in T would be a dominating family for Y of degree less than the degree of W_Y, a contradiction. Therefore $-K_X \cdot T \geq n - 1$ and

$$-K_Z \cdot T_Z = -K_X \cdot T^* \geq -K_X \cdot T + i_X \geq n + 1,$$

so $Z \simeq \mathbb{P}^n$ by Corollary 3.2 and T_Z is the family of lines in Z.

Step 6 *Conclusion.*

Take $l_\sigma - 2$ general sections $H_i \in |\tau^* \mathcal{O}_{\mathbb{P}^n}(1)|$; their intersection $\mathcal{I}$ is a Fano manifold of dimension five with two blow-up contractions of length two $\sigma_{|\mathcal{I}}\colon \mathcal{I} \to Y'$ and $\tau_{|\mathcal{I}}\colon \mathcal{I} \to \mathbb{P}^5$.
By the classification in [**11**] two cases are possible: either the center of $\tau_{|\mathcal{I}}$ is a Veronese surface or it is a cubic scroll contained in a hyperplane. The first case can be excluded noting that, in our case, the degree of E on a minimal curve in R_τ is one, since $E \cdot W = 0$ and $E \cdot R_\sigma = -1$.

It follows that Y' is a del Pezzo manifold of degree five; Y has Y' as an ample section, and therefore Y is a del Pezzo manifold of degree five by repeated applications of [**17**, Proposition A.1]. The only del Pezzo manifold of degree five and dimension greater than five is $\mathbb{G}(1, 4)$. $\square$

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